MATH 1010 Tutorial Nov. 26th (English III)

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Schedule:

5:30 – 6:05 Tutorial presentation
(5 Questions will be discussed)

6:05 – 6:15 Q & A
(Hosted by: YI, Tianhan)

Remark:

- You can visit yzwang.xyz or Blackboard to download the tutorial notes.

- Math Gym (Faculty TA Q&A Centre) is now open. Please visit mathgym.math.cuhk.edu.hk for more information.

1. Evaluate the indefinite integral:

$$\int 2x \arctan(x)\, dx$$

Integration by parts :

u, v two functions

$$\int u\, dv = uv - \int v\, du$$

Observe that $2x \cdot dx = d(x^2)$.

Then $\int 2x \arctan(x)\, dx = \int \arctan(x)\, d(x^2)$

$$= x^2 \arctan(x) - \int x^2\, d(\arctan(x))$$

$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$

$$\int x^2\, d(\arctan(x)) = \int \frac{x^2}{1+x^2}\, dx$$

$$= \int 1 - \frac{1}{1+x^2}\, dx$$

$$= x - \arctan(x) + C$$

Hence

$$\int 2x \arctan(x)\, dx = x^2 \arctan(x) - x + \arctan(x) + C$$

2. Evaluate the indefinite integral:

$$\int \frac{1}{(x+a)(x+b)}\,dx$$

when (a). $a=b$; and (b). $a \neq b$.

(a) $\int \frac{1}{(x+a)^2}\,dx = -\frac{1}{x+a} + C$

(b) Assume $\frac{1}{(x+a)(x+b)} = \frac{E}{x+a} + \frac{D}{x+b}$

Then $\frac{E(x+b)+D(x+a)}{(x+a)(x+b)} = \frac{1}{(x+a)(x+b)}$

Hence $\begin{cases} E+D=0 \\ Eb+Da=1 \end{cases} \Rightarrow \begin{cases} E=\frac{1}{b-a} \\ D=\frac{1}{a-b} \end{cases}$

$$\int \frac{1}{(x+a)(x+b)}\,dx = \int \frac{E}{x+a} + \frac{D}{x+b}\,dx$$

$$= E \cdot \ln|x+a| + D \cdot \ln|x+b| + C$$

$$= \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C$$

3. Evaluate the indefinite integral:

$$\int \sqrt{6x-x^2}\,dx$$

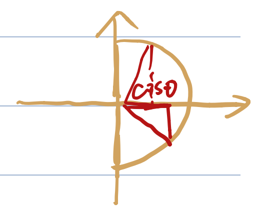
Substitution:

$$\int f(x)\,dx = \int f(u(\theta))\,d(u(\theta))$$
$$= \int f(u(\theta))\cdot u'(\theta)\,d\theta$$

$$\sqrt{6x-x^2} = \sqrt{9-(x^2-6x+9)} = \sqrt{9-(x-3)^2}$$

Hence $(x-3)^2 \le 9$, i.e. $-3 \le x-3 \le 3$.

Let $x-3 = 3\sin\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



$$\int \sqrt{6x-x^2}\,dx = \int \sqrt{9-(x-3)^2}\,dx$$

$$= \int \sqrt{9-9\sin^2\theta}\,d(3\sin\theta)$$
$$= 9\int \sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$$
$$= 9\int \cos^2\theta\,d\theta \quad \leftarrow \quad (1-\sin^2\theta = \cos^2\theta)$$

$$= 9\int \frac{\cos(2\theta)+1}{2}\,d\theta \quad (2\cos^2\theta - 1 = \cos(2\theta))$$

$$= \frac{9}{2}\int \cos(2\theta)\,d\theta + \frac{9}{2}\theta$$

$$= \frac{9}{4}\int \cos(2\theta)\,d(2\theta) + \frac{9}{2}\theta$$

$$= \frac{9}{4}\sin(2\theta) + \frac{9}{2}\theta + C$$

$$\frac{x-3}{3} = \sin\theta \qquad \sqrt{1-\left(\frac{x-3}{3}\right)^2} = \cos\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\int \sqrt{6x-x^2}\,dx = \frac{9}{2}\left(\frac{x-3}{3}\cdot\sqrt{1-\left(\frac{x-3}{3}\right)^2}\right.$$

$$\left.+\arcsin\left(\frac{x-3}{3}\right)\right) + C$$

4. Evaluate the indefinite integral:

$$\int e^x \sin(x)\,dx$$

Since $e^x\,dx = d(e^x)$,

$$\int e^x \sin(x)\,dx = \int \sin(x)\,d(e^x)$$
$$= e^x \cdot \sin(x) - \int e^x\,d(\sin(x))$$
$$= e^x \cdot \sin(x) - \int e^x \cos(x)\,dx$$

Similarly,

$$\int e^x \cos(x)\,dx = \int \cos(x)\,d(e^x)$$
$$= e^x \cdot \cos(x) - \int e^x\,d(\cos(x))$$
$$= e^x \cdot \cos(x) + \int e^x \sin(x)\,dx$$

$$\int e^x \sin(x)\,dx = e^x(\sin(x) - \cos(x)) - \int e^x \sin(x)\,dx$$

Therefore,

$$\int e^x \sin(x)\,dx = \frac{e^x}{2}(\sin(x) - \cos(x)) + C$$

Remark:

$$\int e^x \sin(x)\,dx + \int e^x \sin(x)\,dx = e^x(\sin(x) - \cos(x)) - \int e^x \sin(x)\,dx + \int e^x \sin(x)\,dx$$

$$2 \cdot \int e^x \sin(x)\,dx = e^x(\sin(x) - \cos(x)) + \int 0\,dx$$

$$\int e^x \sin(x)\,dx = \frac{e^x}{2}(\sin(x) - \cos(x)) + C$$

5. Evaluate the indefinite integral:

$$\int \sin^4(x)\,dx$$

given that

$$\int \sin^2(x)\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

and

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

$$\int \sin^4(x)\,dx = -\frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{4}\int \sin^2(x)\,dx$$

$$= -\frac{1}{4}\cos(x)\sin^3(x)$$

$$+\frac{3}{4}\left(\frac{x}{2} - \frac{\sin(2x)}{4}\right) + C$$

$$= -\frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{8}x - \frac{3}{16}\sin(2x) + C$$

"elementary"

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \sin(x)\,dx = -\cos x + C$$