MATH 1010 Tutorial Nov. 19th (English III)

Presenter: WANG, Yizi
Email: yzwang@math.cuhk.edu.hk

Schedule:

5:30 – 6:05 Tutorial presentation
(5 Questions will be discussed)

6:05 – 6:15 Q & A
(Hosted by: YI, Tianhan)

Remark:

- You can visit yzwang.xyz or Blackboard to download the tutorial notes.

- MathGym (Faculty TA Q&A Centre) is now open. Please visit mathgym.math.cuhk.edu.hk for more information.

Q1. Find the first four Taylor polynomials for $f(x) = \ln(x+2)$ at $x = -1$.

$f(-1) = 0 \qquad f'(-1) = \frac{1}{x+2}\Big|_{x=-1} = 1$

$f''(-1) = \left(-\frac{1}{(x+2)^2}\right)\Big|_{x=-1} = -1$

$f^{(3)}(-1) = \left(\frac{2}{(x+2)^3}\right)\Big|_{x=-1} = 2$

$p_0(x) = f(-1) = 0$

$p_1(x) = f(-1) + f'(-1)(x+1) = x+1$

$p_2(x) = p_1(x) + \frac{f''(-1)}{2!}(x+1)^2 = x+1-\frac{1}{2}(x+1)^2$

$p_3(x) = p_2(x) + \frac{f^{(3)}(-1)}{3!}(x+1)^3 = x+1-\frac{1}{2}(x+1)^2+\frac{1}{3}(x+1)^3$

Fact: $\ln(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

Taylor polynomial at $x=0$

$\ln(x+2) = \ln((x+1)+1)$

$= (x+1) - \frac{1}{2}(x+1)^2 + \frac{1}{3}(x+1)^3 - \cdots$

Q2. Let $f(x) = \begin{cases} \dfrac{\cos(2x^3) - 1}{x^3} & \text{otherwise} \\ 0 & \text{if } x = 0 \end{cases}$, f is smooth.

Find $f^{(9)}(0)$.

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$

$$\cos(2x^3) = 1 - \frac{1}{2!}(2x^3)^2 + \frac{1}{4!}(2x^3)^4 - \cdots$$

$$f(x) = \frac{-\frac{2^2}{2!}x^6 + \frac{1}{4!}\cdot 2^4 \cdot x^{12}}{x^3} + \cdots$$

$$= -2x^3 + \frac{2}{3}\cdot x^9 + \cdots$$

$$\frac{f^{(9)}(0)}{9!} = \frac{2}{3}$$

$$f^{(9)}(0) = \frac{2}{3}\cdot 9!$$

Q3. Find the local quadratic approximation of $f(x) = \sqrt{x}$ at $x = 1$.

$f(1) = 1.$ $\quad f'(1) = \frac{1}{2}x^{-\frac{1}{2}}\Big|_{x=1} = \frac{1}{2}$

$f''(1) = -\frac{1}{4}x^{-\frac{3}{2}}\Big|_{x=1} = -\frac{1}{4}$

$$f(x) \approx f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$$

Q4. Calculate the following antiderivatives:

(a) $\int \frac{1}{x} dx$ (b) $\int -2e^x dx$

(c) $\int (2\sin x - \cos x)\, dx$

(a) $\frac{1}{x}$ is defined for $\mathbb{R} \setminus \{0\}$

$\frac{d}{dx} \ln|x| = \frac{1}{x}$ and $\ln|x|$ is defined for $\mathbb{R} - \{0\}$

Hence $\int \frac{1}{x} dx = \ln|x| + C$

(b) $\frac{d}{dx} e^x = e^x$

Then $\int -2e^x dx = -2e^x + C$

(c) $\frac{d}{dx} \sin x = \cos x$ $\frac{d}{dx}(-\cos x) = \sin x$

$\int (2\sin x - \cos x)\, dx = 2\int \sin x\, dx - \int \cos x\, dx$

$= -2\cos x - \sin x$

Q5. Find $f(x)$ where $f''(x) = 2x + \sin(x)$. $f(0) = 1$ and $f'(0) = 0$.

$$f'(x) = \int (2x + \sin x)\,dx = x^2 - \cos x + C_1$$

Since $f'(0) = 0$, $0 - 1 + C_1 = 0$

$C_1 = 1$.

Hence $f'(x) = x^2 - \cos x + 1$.

$$f(x) = \int (x^2 - \cos x + 1)\,dx$$

$$= \frac{x^3}{3} - \sin x + x + C_2$$

Since $f(0) = 1$, then $C_2 = 1$.

Therefore, $f(x) = \dfrac{x^3}{3} - \sin x + x + 1$.