MATH 1010 Tutorial Nov. 12th (English III)

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Schedule:

5:30 – 6:05 Tutorial presentation
(4 Questions will be discussed)

6:05 – 6:15 Q & A
(Hosted by: YI, Tianhan)

Remark:

- You can visit yzwang.xyz or Blackboard to download the tutorial notes.

- MathGym (Faculty TA Q&A Centre) is now open. Please visit mathgym.math.cuhk.edu.hk for more information.

1. Please answer the following questions about the function

$$f(x) = -\frac{2x^2}{x^2-4}.$$

(a) Calculate the first derivative of f. Find the critical numbers of f, where it is increasing and decreasing, and its local extrema.

$f'(x) = $ $\frac{16x}{(x^2-4)^2}$

Critical numbers $x = $ 0

Union of the intervals where $f(x)$ is increasing $[0, 2) \cup (2, +\infty)$

Union of the intervals where $f(x)$ is decreasing $(-\infty, -2) \cup (-2, 0]$

f is continuous at 0

Local maxima $x = $ NONE

Local minima $x = $ 0

$$f'(x) = -\frac{4x(x^2-4) - 2x^2 \cdot 2x}{(x^2-4)^2}$$

$$= \frac{16x}{(x^2-4)^2}$$

$f(x)$ is not defined for $x=2$ or $x=-2$.

(b) Find the following left- and right-hand limits at the vertical asymptote $x = -2$.

$$f(x) = -\frac{2x^2}{x^2-4} = -\frac{2x^2}{(x+2)(x-2)} = \frac{-2x^2}{x-2} \cdot \frac{1}{x+2} \quad \& \quad \lim_{x \to -2} \frac{-2x^2}{x-2} = \frac{-2 \cdot 4}{-2-2} = 2 > 0$$

$$\lim_{x \to -2^-} -\frac{2x^2}{x^2-4} = -\infty \qquad \lim_{x \to -2^+} -\frac{2x^2}{x^2-4} = +\infty$$

Hence $\lim_{x \to -2^-} f(x) = -\infty$ $\qquad \lim_{x \to -2^+} f(x) = +\infty$

Find the following limits at infinity to determine any horizontal asymptotes.

$$\lim_{x \to -\infty} -\frac{2x^2}{x^2-4} = -2 \qquad \lim_{x \to +\infty} -\frac{2x^2}{x^2-4} = -2$$

$$f(x) = -\frac{2}{1 - \frac{4}{x^2}}$$

Hence $\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = -2$

$f'(x) = \dfrac{16x}{(x^2-4)^2}$

(c) Calculate the second derivative of f. Find where f is concave up, concave down, and has inflection points.

$f''(x) =$ $\dfrac{-16(3x^2+4)}{(x^2-4)^3}$

Union of the intervals where $f(x)$ is concave up $(-2, 2)$

Union of the intervals where $f(x)$ is concave down $(-\infty, -2) \cup (2, +\infty)$

Inflection points $x =$ NONE

$$f''(x) = 16\left[\frac{(x^2-4)^2 - x \cdot 2(x^2-4) \cdot 2x}{(x^2-4)^4}\right]$$

$$= 16\left[\frac{x^2-4-4x^2}{(x^2-4)^3}\right] = -16 \cdot \frac{3x^2+4}{(x^2-4)^3}$$

$f''(x) > 0$ if $x^2 < 4$

$f''(x) < 0$ if $x^2 > 4$

(d) The function f is even/~~odd~~ because $f(x) = f(-x)$ for all x in the domain of f, and therefore its graph is symmetric about the y-axis

$$f(x) = -\frac{2x^2}{x^2-4} = f(-x)$$

(e) Answer the following questions about the function f and its graph.

The domain of f is the set (in **interval notation**) $(-\infty, -2) \cup (-2, 2) \cup (2, +\infty)$

The range of f is the set (in **interval notation**) $(-\infty, -2) \cup [0, +\infty)$

y-intercept $y = 0$

x-intercepts $x = 0$

Let $f(x) = y$

$$-\frac{2x^2}{x^2-4} = y$$

$$-2x^2 = y(x^2-4)$$

$$(y+2)x^2 - 4y = 0$$

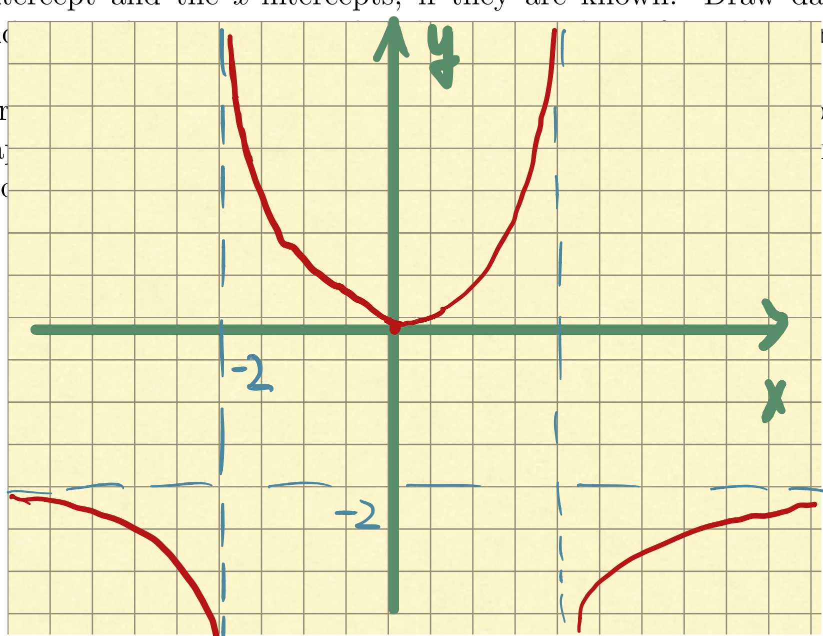
$$(y+2)x^2 = 4y \quad (*)$$

If $-2 \le y < 0$, $\nexists\ x$ satisfies $(*)$

If $y < -2$, $x = \sqrt{\dfrac{4y}{y+2}}$

If $y \ge 0$, $x = \sqrt{\dfrac{4y}{y+2}}$

(f) Sketch a graph of the function f without having a graphing calculator do it for you. Plot the y-intercept and the x-intercepts, if they are known. Draw dashed lines for horizontal an... maxima, local minima, and ... to sketch the remaining par... our advantage. Sketching gra... be asked to do it on quizzes o...



2. Suppose that

$$f(x) = (x-1)^2 \ln(x-1), \quad x > 1.$$

(A) List all the critical values of $f(x)$. Note: If there are no critical values, enter 'NONE'.

$e^{-\frac{1}{2}}+1$

$$f'(x) = 2(x-1)\ln(x-1) + (x-1) = (x-1)[2\ln(x-1)+1]$$

$$e^{2\ln(x-1)+1} = 1 \Rightarrow (x-1)^2 \cdot e = 1$$

(B) Use interval notation to indicate where $f(x)$ is increasing.
Increasing: $[e^{-\frac{1}{2}}+1, +\infty)$

(C) Use interval notation to indicate where $f(x)$ is decreasing.
Decreasing: $(1, e^{-\frac{1}{2}}+1]$

(D) List the x values of all local maxima of $f(x)$.
x values of local maximums = NONE

(E) List the x values of all local minima of $f(x)$.
x values of local minimums = $e^{-\frac{1}{2}}+1$

(F) Use interval notation to indicate where $f(x)$ is concave up.
Concave up: $(e^{-\frac{3}{2}}+1, +\infty)$

(G) Use interval notation to indicate where $f(x)$ is concave down.
Concave down: $(1, e^{-\frac{3}{2}}+1)$

$$f''(x) = \frac{d}{dx}(x-1)[2\ln(x-1)+1]$$

$$= 2\ln(x-1)+1 + (x-1)\cdot\frac{2}{x-1}$$

$$= 2\ln(x-1)+3$$

$$f''(x) = 0 \iff e^{2\ln(x-1)+3} = 1$$

$$(x-1)^2 \cdot e^3 = 1$$

$$x = e^{-\frac{3}{2}}+1$$

3. Compute

$$\lim_{x\longrightarrow 0} \frac{\overset{\to 0}{e^{2x} - e^{-2x}}}{\underset{\downarrow 0}{2\sin x}} = \underline{\qquad\qquad}$$

M1: " $\frac{0}{0}$ "

$$\frac{d}{dx}(e^{2x} - e^{-2x}) = 2e^{2x} + 2e^{-2x}$$

$$\frac{d}{dx}(2\sin x) = 2\cos x$$

$$\lim_{x\to 0} \frac{2e^{2x} + 2e^{-2x}}{2\cos x} = \frac{4}{2} = 2$$

M2: " $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ "

$$\lim_{x\to 0} \frac{e^{2x} - e^{-2x}}{x} \cdot \frac{x}{2\sin x}$$

$$= \lim_{x\to 0} \underbrace{\frac{e^{4x} - 1}{4x}}_{\downarrow\, 1} \cdot \underbrace{\frac{x}{2\sin x}}_{\downarrow\, \frac{1}{2}} \cdot \underbrace{\frac{4}{e^{2x}}}_{\downarrow\, 4}$$

$$= 2$$

"$0 \cdot \infty$"

4. Use L'Hôpital's Rule (possibly more than once) to evaluate the following limit

$\lim_{t\to 0} \sin(t)\ln(t) =$ ________

M1: $$\lim_{t\to 0} \frac{\ln(t)}{\frac{1}{\sin(t)}} \overset{(H)}{=} \lim_{t\to 0} \frac{\frac{1}{t}}{\frac{-\cos(t)}{(\sin(t))^2}}$$

$$= \lim_{t\to 0} \frac{(\sin(t))^2}{-t\cos(t)} \to 0$$

$$= \lim_{t\to 0} \underbrace{\frac{\sin(t)}{t}}_{\to 1} \cdot \underbrace{\frac{\sin(t)}{-\cos(t)}}_{\to 0}$$

$$\overset{(H)}{=} \lim_{t\to 0} \frac{2\sin(t)\cos(t)}{-\cos(t) + t\sin(t)}$$

$$= 0$$

M2: "$\lim_{x\to +\infty} \frac{\ln x}{x} = 0$"

Hence $$\sin t \ln t = t \cdot \ln(t) \cdot \frac{\sin(t)}{t}$$

$(x = \frac{1}{t})$

$$\lim_{t\to 0} t\cdot \ln t = \lim_{x\to +\infty} \frac{1}{x} \cdot \ln\left(\frac{1}{x}\right)$$

$$= \lim_{x\to +\infty} - \frac{\ln(x)}{x}$$

$$= 0$$

$$\lim_{t\to 0} \sin t \cdot \ln(t) = \left(\lim_{t\to 0} t \cdot \ln(t)\right) \cdot \left(\lim_{t\to 0} \frac{\sin(t)}{t}\right) = 0$$