# MATH 1010 Tutorial Oct. 15th
# (English III)

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Schedule:

5:30 – 6:05 Tutorial presentation
(5 Questions will be discussed)

6:05 – 6:15 Q & A
(Hosted by: YI, Tianhan)

Remark:

- You can visit yzwang.xyz or Blackboard to download the tutorial notes.

- Math Gym (Faculty TA Q&A Centre) is now open. Please visit mathgym.math.cuhk.edu.hk for more information.

Q1: For what value of $c$ is the function $f$ continuous on $(-\infty, \infty)$ where

$$f(x) = \begin{cases} 12x - 10 & x \leq c \\ 2x^2 + 8 & x > c \end{cases}$$

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^-} 12x - 10 = 12c - 10 = f(c)$$

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^+} 2x^2 + 8 = 2c^2 + 8$$

$f$ continuous at $c$

$$\Leftrightarrow \quad 2c^2 + 8 = 12c - 10$$

$$\Leftrightarrow \quad c^2 - 6c + 9 = 0$$

Hence $c = 3$

Q2: (a) Let $f(x) = |x-2|$.

Determine whether $f$ is differentiable at 2.

① Continuity

$$\lim_{x \to 2^-} |x-2| = \lim_{x \to 2^+} |x-2| = 0 = f(2)$$

② Differentiability

$$\frac{f(2+h) - f(2)}{h} = \frac{|2+h-2| - 0}{h} = \frac{|h|}{h}$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = -1 \neq \lim_{h \to 0^+} \frac{|h|}{h} = 1$$

(b) Let $g(x) = \begin{cases} 3x^2 + 4x & \text{if } x < 0 \\ x + 2 & \text{if } x \geq 0 \end{cases}$

Determine whether $g$ is differentiable at 0.

Since $\lim_{x \to 0^-} g(x) = 0$ and $\lim_{x \to 0^+} g(x) = 2$,

$g$ is not continuous at 0.

Hence $g$ is not differentiable at 0.

Q3: For what value of $a$ is the function $f$ continuous on $(-\infty, \infty)$ where

$$f(x) = \begin{cases} \dfrac{3x^3 + 2x^2 - 4x - 1}{x-1} & \text{if } x < 1 \\ 4x^2 + x + a & \text{if } x \geq 1 \end{cases}$$

## Long division with polynomial:

$$\begin{array}{r} 3x^2 + 5x + 1 \\ x - 1 \,\big)\, \overline{3x^3 + 2x^2 - 4x - 1} \\ -(3x^3 - 3x^2) \\ \hline 5x^2 - 4x - 1 \\ -(5x^2 - 5x) \\ \hline x - 1 \\ x - 1 \\ \hline 0 \end{array}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3x^2 + 5x + 1 = 9$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4x^2 + x + a = 5 + a = f(1)$$

$f$ continuous if and only if $9 = 5 + a$ which is $a = 4$.

Q4. Let $f(x) = e^{3x}$.

Use the limit definition of the derivative to find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{3(x+h)} - e^{3x}}{h}$$

$e^{a+b} = e^a \cdot e^b$

$$= \lim_{h \to 0} \frac{e^{3x} \cdot e^{3h} - e^{3x}}{h}$$

$$= e^{3x} \lim_{h \to 0} \frac{e^{3h} - 1}{h}$$

$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$$= 3e^{3x} \lim_{h \to 0} \frac{e^{3h} - 1}{3h}$$

$$= 3e^{3x}$$

Q5. Let $f(x) = \begin{cases} x^2\cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Use the limit definition of the derivative to find $f'(x)$.

Case 1: $x = 0$

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2\cos(\frac{1}{h})}{h}$$

$$= \lim_{h\to 0} h\cos(\frac{1}{h})$$

$= 0$ by squeeze theorem

Case 2: $x \neq 0$

$$\frac{f(x+h) - f(x)}{h}$$

$$= \frac{1}{h}\left[(x+h)^2\cos(\frac{1}{x+h}) - x^2\cos(\frac{1}{x})\right]$$

$$= \frac{1}{h}\left[(x^2+2xh+h^2)\cos(\frac{1}{x+h}) - x^2\cos(\frac{1}{x})\right]$$

$\to 2x\cos(\frac{1}{x})$

$$= \underline{(2x+h)\cos(\frac{1}{x+h})} + x^2\frac{1}{h}\left[\cos(\frac{1}{x+h}) - \cos(\frac{1}{x})\right]$$

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

$$\frac{1}{h}\left[\cos\left(\frac{1}{x+h}\right) - \cos\left(\frac{1}{x}\right)\right]$$

$$\frac{x-(x+h)}{(x+h)x}$$

$$= \frac{1}{h}\left[-2\sin\left(\frac{\frac{1}{x+h}+\frac{1}{x}}{2}\right)\sin\left(\frac{\frac{1}{x+h}-\frac{1}{x}}{2}\right)\right]$$

$$= \underbrace{-2\sin\left(\frac{\frac{1}{x+h}+\frac{1}{x}}{2}\right)}_{\downarrow\; -2\sin\left(\frac{1}{x}\right)} \cdot \frac{1}{h}\sin\left(\frac{-h}{2(x+h)x}\right)$$

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1.$$

$$\lim_{h\to 0}\frac{1}{h}\sin\left(\frac{-h}{2(x+h)x}\right)$$

$$= \lim_{h\to 0}\left(-\frac{1}{2(x+h)x}\right)\cdot\frac{\sin\left(\frac{-h}{2(x+h)x}\right)}{\frac{-h}{2(x+h)x}}$$

$$= -\frac{1}{2x^2}$$

Hence $f'(x) = 2x\cos\left(\frac{1}{x}\right) + x^2\left(-2\sin\left(\frac{1}{x}\right)\right)\left(-\frac{1}{2x^2}\right)$

$$= 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

$\square$