MATH 1010 Tutorial Sep. 17th (English III)

Presenter: WANG, Yizi

Email: yzwang@math.cuhk.edu.hk

Schedule:

5:30 – 6:05 Tutorial presentation

(5 Questions will be discussed)

6:05 – 6:15 Q & A

(Hosted by: YI, Tianhan)

Q1.

Determine whether the sequence

$$a_n = \frac{n^7 + \cos(6n^2+2)}{n^9}$$

converges or diverges.
If it converges, find the limit.

Sol: $-1 \leq \cos(6n^2+2) \leq 1$

$$b_n := \underbrace{\frac{n^7-1}{n^9}}_{\to 0} \leq a_n \leq \underbrace{\frac{n^7+1}{n^9}}_{\to 0} =: c_n$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{1}{n^2} - \frac{1}{n^9}}{1} = 0$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\frac{1}{n^2} + \frac{1}{n^9}}{1} = 0.$$

By squeeze theorem, $\lim_{n\to\infty} a_n = 0$.

Q2.

Determine whether the sequence

$$a_n = \frac{7+\arctan(n^n+4)}{2^n}$$

converges or diverges.

If it converges, find the limit.

Sol: $-\frac{\pi}{2} \le \arctan(n^n+4) \le \frac{\pi}{2}$

$$\frac{7-\frac{\pi}{2}}{2^n} \le a_n \le \frac{7+\frac{\pi}{2}}{2^n}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$0 \qquad\qquad\qquad 0$$

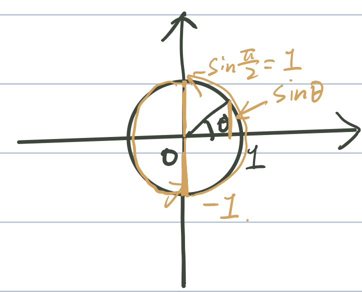
By squeeze theorem, $\lim_{n\to\infty} a_n = 0$

Q3.

Consider the sequence

$$a_n = \frac{2n\sin(\frac{\pi}{2}+n\pi)}{3n+2}$$

Write the first five terms of a_n, and find $\lim_{n\to\infty} a_n$ if it exists.



Sol: $\sin(\frac{\pi}{2}+n\pi) = ?$

$\sin(\frac{\pi}{2}+n\pi) = (-1)^n$.

$a_1 = -\frac{2}{5}$ $\quad a_2 = \frac{4}{8} = \frac{1}{2}$ $\quad a_3 = -\frac{6}{11}$

$a_4 = \frac{8}{14} = \frac{4}{7}$ $\quad a_5 = -\frac{10}{17}$

$b_n = \frac{2n}{3n+2} \to \frac{2}{3}$ as $n\to\infty$

When $n=2k$ for $k\in\mathbb{N}$, $a_{2k} = \frac{2(2k)}{3(2k)+2}$.

$\lim_{k\to\infty} a_{2k} = \frac{2}{3}$.

When $n=2k+1$, $a_{2k+1} = -\frac{2(2k+1)}{3(2k+1)+2}$.

$\lim_{k\to\infty} a_{2k+1} = -\frac{2}{3}$

Since $\frac{2}{3} \neq -\frac{2}{3}$, then $\lim_{n\to\infty} a_n$ does not exist.

Q4.

Consider the sequence $a_n = \frac{1}{n^2+n}$.

(a) Find $\lim_{n\to\infty} a_n$ if it exists.

(b) Find $\sum_{n=1}^{\infty} a_n$ if it exists.

Sol: (a) $\lim_{n\to\infty} \frac{1}{\underbrace{n^2+n}_{\to\infty}} = 0$

(b) Let $S_n = \sum_{k=1}^{n} a_k$.

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

$a_n = \frac{1}{n^2+n} = \frac{1}{(n+1)n} = \frac{1}{n} - \frac{1}{n+1}$

Hence $S_n = \sum_{k=1}^{n} a_k$

$= (1 - \cancel{\tfrac{1}{2}}) + (\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}) \cancel{+} \cdots \cancel{+} (\cancel{\tfrac{1}{n}} - \tfrac{1}{n+1})$

$= 1 - \underbrace{\frac{1}{n+1}}_{\to 0}$

$\lim_{n\to\infty} S_n = 1$.

Q5.

Consider the recursively defined sequence

$a_1 = 2\sqrt{3}$, $a_{n+1} = \sqrt{12 + a_n}$ for $n \geq 1$

Determine if the sequence defined above converges or diverges and find its limit if it converges.

Ideas: Monotone convergence theorem

$\underbrace{\text{Bounded}}_{\text{I}} + \underbrace{\text{Monotonic}}_{\text{II}} \Rightarrow \text{Convergent.}$

Use Mathematical Induction (M.I.)

II: Let $P(n)$ be the statement that "$a_{n+1} \geq a_n$".

Goal: $P(n)$ is true for all $n \geq 1$.

① When $n = 1$.

$a_2 = \sqrt{12 + a_1} = \sqrt{12 + 2\sqrt{3}} \geq \sqrt{12} = a_1$

Hence $P(1)$ is true.

② We assume $P(n)$ is true for some $n \geq 1$.

i.e. $a_{n+1} \geq a_n$

Then $a_{n+2} = \sqrt{12 + a_{n+1}}$ $\qquad a_{n+1} = \sqrt{12 + a_n}$

$\geq \sqrt{12 + a_n} = a_{n+1}$

Hence $P(n+1)$ is true.

Therefore a_n is increasing.

I. Lower bound: $a_1 = 2\sqrt{3}$ $\Rightarrow$ $a_n \geq 2\sqrt{3}$

Upper bound: Let $P(n)$ be the statement that "$a_n \leq 4$".

① $a_1 = 2\sqrt{3} = \sqrt{12} \leq \sqrt{16} = 4$

$P(1)$ is true

② Assume $P(n)$ is true for $n \geq 1$, i.e. $a_n \leq 4$.

$a_{n+1} = \sqrt{12 + a_n} \leq \sqrt{12+4} = \sqrt{16} = 4$.

Hence $a_n \leq 4$ for $n \geq 1$.

Limit: Let $\lim_{n\to\infty} a_n = L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{12 + a_n}$$

$$L = \sqrt{12 + L}$$

$$L^2 - L - 12 = 0$$

$L = 4$ or ($L = -3$) dropped.

Hence $\lim_{n\to\infty} a_n = 4$.